Synthesis of Integrated Process Designs," *Ind. Eng. Chem. Fundamentals,* **9,** 48 (1970).
Lingafelter, T. E., and S. Lynn, "Sulfur Abatement from Power Plant Stack Gas Through Coal Gasification," CP-10 in *Case Problems in Chemical Process Design,* Univ. California, Berkeley (1971).
Lockhart, F. J., "Multi-Column Distillation of Natural Gasoline," *Petrol. Refiner,* **26,** 104 (1947).
Masso, A. H., "Studies in Process Engineering," Ph.D. dissertation, Univ. of Wisconsin, Madison (1968).
———., and D. F. Rudd, "The Synthesis of System Designs, II: Heuristic Structuring," *AIChE J.,* **15,** 10 (1969).
McGalliard, R. L., and A. W. Westerberg, "Structural Sensitivity Analysis in Design Synthesis," paper presented at Am. Inst. Chem. Engrs. Meeting, Dallas, Texas (1972).
Meissner, H. P., *Processes and Systems in Industrial Chemistry,* Prentice-Hall, Englewood Cliffs, N. J. (1971).
Menzies, M. A., and A. I. Johnson, "Synthesis of Optimal Energy Recovery Networks Using Discrete Methods," *Can J. Chem. Eng.,* **50,** 290 (1972a).
———., "Synthesis of Optimal Energy Recovery Networks Using Discrete Methods," paper presented at Am. Inst. Chem. Engrs. Meeting, Dallas, Texas (1972).
Middendorf, W. H., *Engineering Design,* Allyn and Bacon, Boston (1969).
Miller, C. E., "The Simplex Method for Local Separable Programming," in *Recent Advances in Mathematical Programming,* McGraw-Hill, New York (1963).
Mitten, L. G., and G. L. Nemhauser, "Multistage Optimization," *Chem. Eng. Progr.,* **59,** No. 1, 52 (1963).
Nishida, N., S. Kobayashi, and A. Ichikawa, "Optimal Synthesis of Heat Exchange Systems, Necessary Conditions for Minimum Heat Transfer Area and Their Application to Systems Synthesis," *Chem. Eng. Sci.,* **26,** 1841 (1971).
Nishimura, H., and Y. Hiraizumi, "Optimal System Pattern for Multicomponent Distillation Systems," *Intern. Chem. Eng.,* **11,** 188 (1971).
Oliver, E. D., *Diffusional Separation Processes: Theory, Design, and Evaluation,* Wiley, New York (1966).
Petlyuk, F. B., V. M. Platonov, and D. M. Slavinskii, "Thermodynamically Optimal Method for Separating Multicomponent Mixtures," *Int. Chem. Eng.,* **5,** 555 (1965).
Porter, B., *Synthesis of Dynamical Systems,* Barnes and Noble, New York (1969).
Powers, G. J., "Recognizing Patterns in the Synthesis of Chemical Processing Systems," Ph.D. Thesis, Univ. Wisconsin, Madison (1971).
———., "Heuristic Synthesis in Process Development," *Chem. Eng. Prog.,* **68,** No. 8, 88 (1972).
Rod, V., and J. Marek, "Separation Sequences in Multicomponent Rectification," *Coll. Czech, Chem. Comm.,* **24,** 3240 (1959).
Rudd, D. F., "The Synthesis of System Designs, I: Elementary Decomposition Theory," *AIChE J.,* **14,** 343 (1968).
———., G. J. Powers, and J. J. Siirola, *Process Synthesis,* Prentice-Hall, Englewood Cliffs, N. J. (1973).
Rudd, D. F., and C. C. Watson, *Strategy of Process Engineering,* Wiley, New York (1968).
Russell, T. W. F., and D. T. Buzzelli, "Reactor Analysis and Process Synthesis, *Ind. Eng. Chem. Process Design Develop.* **8,** 2 (1969).
Sherwood, T. K., *A Course in Process Design,* The M.I.T. Press, Cambridge, Mass. (1963).
Siirola, J. J., "The Computer-Aided Synthesis of Chemical Process Designs," Ph.D. thesis, Univ. Wisconsin, Madison (1970).
———., G. J. Powers, and D. F. Rudd, "Synthesis of System Designs, III: Toward a Process Concept Generator," *AIChE J.,* **17,** 677 (1971).
Siirola, J. J., and D. F. Rudd, "Computer-Aided Synthesis of Chemical Process Designs," *Ind. Eng. Chem. Fundamentals,* **10,** 353 (1971).
Smith, B. D., et al., "Design Case Studies," Dept. of Chem. Eng., Washington Univ., St. Louis, Missouri (1967-1971).
Stupin, W. J., and F. J. Lockhart, "Thermally Coupled Distillation," Reprint 64th Annual AIChE Meeting (1971).
Ten Broeck, H., "Economic Selection of Exchanger Sizes," *Ind. Eng. Chem.,* **36,** 64 (1944).
Thompson, R. W., and C. J. King, "Desalinization by Reverse Osmosis," CP-4 in *Case Problems in Chemical Process Design,* Univ. California, Berkeley (1971).
———., "Systematic Synthesis of Separation Schemes," paper presented at Am. Inst. Chem. Engrs. Meeting, Dallas, Texas (1972).
Umeda, T., A. Hirai, and A. Ichikawa, "Synthesis of Optimal Processing System by an Integrated Approach," *Chem. Eng. Sci.,* **27,** 795 (1972).
Umeda, T., and A. Ichikawa, "Synthesis of Optimal Processing Systems by a Method of Decomposition," paper presented at Am. Inst. Chem. Engrs. Meeting, Dallas, Texas (1972).
Westbrook, G. T., "Use This Method to Size Each Stage for Best Operation," *Hydrocarbon Processing Petrol. Refiner,* **40,** 201 (1961).

# Determination of Friction Factors for Pulsatile Flow of Water in Distensible Tubes

**Theoretical models for computing velocity profiles and friction factors for sinusoidally pulsing laminar flow in linearly elastic tubes were derived from the Navier-Stokes equations, a modified Fanning equation, and the pressure-tube wall movement relationship. The theoretical results were checked experimentally for $N_{Re}$ and $\lambda = \omega R^2/\nu$ values up to 2000 and 57, respectively. It was found that the theoretical friction factors predicted the experimental values to within less than 13%. Also the results showed that the energy losses in elastic tubes were greater than in rigid tubes.**

**JOHN S. AHN and DANIEL HERSHEY**

**Department of Computer Science**
**University of Utah**
**Salt Lake City, Utah 84112**

## SCOPE

**A study of pulsating flow in distensible tubes has received attention from investigators interested in research on the human circulatory system. Much effort has been directed towards describing blood flow satisfactorily. This requires a thorough knowledge of the tapering and branching of the blood vessels, mechanical properties of the ves-**

D. Hershey is with the Department of Chemical Engineering, University of Cincinnati, Cincinnati, Ohio 45221.

sels, non-Newtonian behavior of the blood, changes in pulse wave shapes, and other factors.

Simplified versions of the Navier-Stokes equations have been used to describe pulsatile flow since it is difficult to solve the original equations. This is usually done by assuming the flow to be either inertial or viscous in nature. Pulsatile flow phenomena in rigid tubes have been studied by solving the simplified equations either analytically or numerically. Experimental results have supported the validity of some of the analytical solutions.

There were two primary objectives in the present work. First, a mathematical model describing the pulsating flow phenomena in distensible tubes was derived from the Navier-Stokes equations. A pressure-tube wall displacement relationship was developed and coupled with the flow equations, which were solved by the Laplace transform technique. The resulting expressions for the velocity profile and friction factor reduced to expressions for the steady state flow in a distensible tube when the frequency was set equal to zero.

Secondly, experimental equipment was set up in order to check the validity of the theoretical equations. The flow range of the experiment was $300 \leq N_{Re} \leq 2000$ and $0 \leq \lambda\ (= \omega R^2/\nu) \leq 57$, where $\lambda$ is a dimensionless frequency parameter. The pressure pulse was measured with strain gauges attached to the surface of the tube.

## CONCLUSIONS AND SIGNIFICANCE

With the derived analytical solution for pulsatile flow in a distensible tube, the velocity and volumetric flow rate can now be predicted. The theoretical and experimental results agreed to within 13% for $\lambda$ values up to 20. For $\lambda$ above 20, the theoretical and experimental results diverged. It is believed for $\lambda$ above 20 that the effect of fluid acceleration and deceleration dominates over the effect of viscosity. However, further experiments are necessary to define this laminar region of the pulsing flow.

Since $\lambda$ values for human femoral arteries are estimated to be around 10, the results of this research can be used in hemodynamic studies. The experimental technique using strain gauges attached peripherally can be applied to the human body.

Both experimental and theoretical results indicate that there is more energy loss in the distensible tube than in the rigid tube.

One-dimensional pulsating flow with a periodic pressure gradient was treated theoretically as early as 1930 by Sexl (1930). Hershey and Song (1967) derived an equation relating the friction factor to the Reynolds number and the frequency parameter $\lambda$ for pulsating flow of a Newtonian fluid in a rigid tube from the Navier-Stokes equation, the equation of continuity and an expression describing the sinusoidal pressure. The equations were solved by the Laplace transform technique with appropriate initial and boundary conditions. Hershey and Song found that the experimental results for water and sheep's blood agreed with the theory within 5% for $\lambda$ values up to 17.

Uchida (1956) and Atabek and Chang (1961) expressed the pulsating flow rate in a rigid tube as sum of the steady state term and the pulsating flow term. Uchida showed by using his theoretical results that there is more energy dissipation for pulsatile flow than for steady state flow at a given average flow rate. Hershey and Song (1967) showed it theoretically as well as experimentally.

Sheppard (1967) and Fairchild et al. (1967) approached the one-dimensional flow problem numerically. Sheppard eliminated the radial flow component by integrating each term of the equation over the cross section of the tube. The equations were solved numerically by a fourth-order Runge-Kutta method using the pressure-diameter relationship $P(R) = E\ \delta\ (1/Ru - 1/R)$ where $E$ is the Young's modulus, $\delta$ is the thickness of the tube, and $Ru$ is the radius of the unstressed tube. His theory was checked experimentally using water and ethylene glycol, and it was found that water agreed better than the more viscous ethylene glycol. By using the AMOS numerical integration method, Fairchild et al. obtained numerical solutions for the unsteady state flow of a Newtonian fluid in a rigid cylindrical tube and the unsteady state flow of an ideal fluid in an elastic tube.

Various two-dimensional flow models have also been constructed from the Navier-Stokes equation of motion, the equation of motion of the tube, the equation of continuity, and the oscillatory pressure expression. Womersley (1957) described the pulsing flow phenomenon in a distensible tube with a sinusoidal pressure gradient expressed by $dP/dz = Ae^{i\omega\tau}$ where $A$ is the amplitude of the pulsation, $\omega$ is angular velocity, and $\tau$ is time. He neglected the nonlinear inertial terms in the Navier-Stokes equation of motion and later estimated the terms from the known solution of the linear differential equations. He coupled the flow equation with the visco-elastic equation of the distensible tube and then solved the equations simultaneously. Solutions by his method involved extensive calculations.

Chang and Atabek (1961) investigated pulsating flow near the entrance of a rigid cylindrical tube. The nonlinear term of the Navier-Stokes equation was linearized by assuming that the axial pulsating velocity at the entrance section was a function of time only. Using his derived pressure versus tube radius relationship for a linearly elastic tube, Lambert (1958) solved the nonsteady flow of an inviscid and incompressible fluid in a nonrigid tube. He reduced the equation to one-dimensional flow by integrating across the tube cross section. The pressure wave form change calculated from his model agreed well qualitatively with that in a dog's aorta. Lambert's results stress the importance of the nonlinear terms of the equation of motion.

Streeter et al. (1963) derived differential equations from the conventional mass and momentum balance of the fluid in a tapered distensible tube. The tube radius-pressure relationship was derived from a simple force balance between the tensile force change of the wall and the fluid pressure in the tube. When the flow rate was computed by substituting a laminar flow friction factor into the numerical solution, the predicted flow was found to be greater than the flow rate measured with an electromagnetic flowmeter. The authors concluded that the frictional energy loss for pulsating flow in a distensible tube is greater than that for Poiseuille flow, confirming the results of Hershey and Song (1967) and Uchida (1956). Auster (1968) numerically solved the flow equations for steady state flow in a tapered rigid tube by using an implicit-finite difference method. The radial velocity was approximated by the stream function and then the flow equation was solved for the axial velocity. Morgan and Ferrante (1955) investigated wave propagation through a streaming liquid in an elastic tube. They solved the problem by a linearized perturbation of the steady state flow. This analysis is restricted to small pulse amplitude and viscosity.

Since the above models (except Streeter et al.) were constructed on the assumption of laminar flow, it is appropriate here to briefly discuss the range of the laminar region. Hershey and Im (1968) defined the laminar transition as the point where the slope of the experimental friction factor versus Reynolds number curve changes abruptly on a log-log plot. They found that the critical Reynolds number for pulsating flow was lower than for Poiseuille flow. They also found that the critical Reynolds number decreased as $\lambda$ increased. Other authors defined the transition point in a different way. When a turbulent slug was created upstream in a smooth flowing fluid, Gilbrech and Combs (1964) defined the growth rate as $G = \dfrac{V_L - V_T}{\overline{V}}$ where $V_L$ is the velocity of the leading edge of the slug, $V_T$ is the velocity of the trailing edge, and $\overline{V}$ is the mean velocity. The Reynolds number, at which the growth rate is zero, was called the critical Reynolds number. From the investigations of Gilbrech and Combs (1964), Yellin (1966), and Sarpkaya (1966), the critical Reynolds number for pulsating flow was found to be a function of the flow amplitude ratio and $\lambda$. The flow amplitude ratio was defined as $\hat{Q}/\overline{Q}$ where $\hat{Q}$ is amplitude of the periodic component of the volume flow and $\overline{Q}$ is the mean volume flow. The critical Reynolds number for pulsating flow was found to be higher than the corresponding steady state flow at high flow amplitude ratios and low values of the frequency parameter $\lambda$. The situation was reversed at low flow amplitude ratios and high values of the frequency parameter. Sarpkaya (1966) attempted to relate instability of oscillatory flow to the point of inflection in the velocity profile.

The work which is being reported upon in this paper extends the work of Hershey and Song (1967) from one-dimensional flow in a rigid tube to two-dimensional flow in a distensible tube. The radial flow component, which was absent in the rigid tube, will now be considered in the nonrigid tube case.

## MATHEMATICAL MODEL

### Assumptions

1. The flow is laminar and the fluid is Newtonian.
2. The tube is linearly elastic, cylindrical, homogeneous in composition, smooth, and of uniform diameter and thickness. Sheppard (1967) and Streeter et al. (1963) showed that there is a linear relationship between fluid pressure and the tube wall radius for small changes of the radius. A rubber tube which meets the above assumptions was used as a test section.
3. The entrance effect is negligible. For the tube size of $4.740 \times 10^{-3}$ m which was used for the experiments, the entrance length was built to be approximately 0.7 m, as determined from $L = 1.6 \times 10^{-3} <R> N_{Re}$ given by Chang and Atabek (1961).
4. The longitudinal distension is neglected.
5. There is no slip at the wall.
6. The pressure pulse upstream is in phase with the downstream pulse. (According to our experimental results, this is true for $\lambda$ values up to 20).
7. Calibration of tube properties at the static condition is applicable to the dynamic condition.

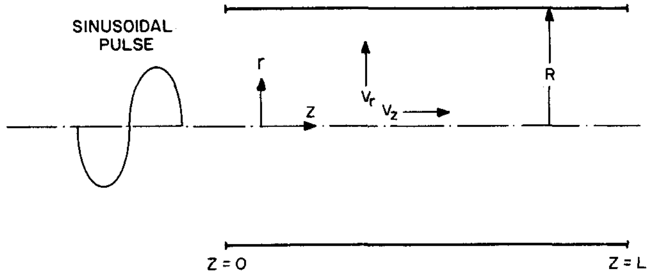


**Fig. 1. Pulsating flow velocities in a distensible tube.**

With the above assumptions the Navier-Stokes equation of motion and the equation of continuity were used to derive a friction factor relationship for sinusoidal flow of an incompressible Newtonian fluid in a distensible tube by a method analogous to that of Hershey and Song (1967). Figure 1 shows the flow situation.

Assuming axial symmetry and no gravitational effect, the Navier-Stokes equation of motion (Ahn, 1970) is

*z-component*

$$\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{1}{\rho}\frac{dP}{dz} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right] \quad (1)$$

*r-component*

$$\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{1}{\rho}\frac{dP}{dr} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] \quad (2)$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho\, r\, v_r) + \frac{\partial}{\partial z}(\rho\, v_z) = 0 \quad (3)$$

The upstream pressure is $P_0(1 + \sin \omega t)$ which was used by Hershey and Song (1967). The downstream pressure, also according to Hershey and Song (1967) is

$$P = (P_0 - Az)\,(1 + \sin \omega t) \quad (4)$$

where

$$A = \frac{f_p\, \rho <v_z>^2}{<R>\, g_c} \quad (5)$$

and $f_p$ is the Fanning friction factor. Equation (4) reduces to the Fanning equation when $\omega$ is zero (steady flow).

In order to obtain a more convenient initial condition as shown later, a transformation of the time variable was made, $\omega t = \omega \tau + 3\,\pi/2$, which gives Equation (6).

$$P = (P_0 - Az)(1 - \cos \omega\tau) \quad (6)$$

The radius of the tube is correlated against the fluid pressure. For small variations of the tube wall radius, up to 4% distension, it has been experimentally determined (Ahn, 1970) that the radial distension depends linearly on pressure as shown in Equation (7)

$$R\,(z, \tau) = R_0 + \beta\, P \quad (7)$$

The radial velocity can be estimated from the definition of the stream function (Ahn, 1970) and applying it to the unsteady state case; the result is Equation (8)

$$v_r = \frac{r}{R}\left(\frac{\partial R}{\partial \tau} + v_z \frac{\partial R}{\partial z}\right) \quad (8)$$

The radial boundary conditions are $r = 0$, $v_r = 0$, and $r = R$, $v_r = \partial R/\partial \tau$. Combining Equation (8) with Equations (6) and (7), the result is Equation (9)

$$v_r = \frac{\beta \omega P_0 - Az}{R} r \sin \omega\tau - \frac{\beta A}{R} v_z r (1 - \cos \omega\tau) \tag{9}$$

Equation (1) is simplified by eliminating the first nonlinear term $v_r \frac{\partial v_z}{\partial r}$. A qualitative comparison between the two nonlinear terms in Equation (1) was made (Ahn, 1970) to show that $v_z \frac{\partial v_z}{\partial z}$ is more important than $v_r \frac{\partial v_z}{\partial r}$. [Various authors have also dropped the term $v_r \frac{\partial v_z}{\partial r}$ while keeping the other nonlinear term (Hanks, 1963; Lambert, 1958)]. With this simplification and Equations (3) and (9), we can get from Equation (1)

$$\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) - \frac{r}{\nu}\frac{\partial v_z}{\partial \tau} = \left(\frac{2 A \beta}{\nu R} r v_z^2 - \frac{Ar}{\mu}\right) (1 - \cos \omega\tau) - \left[\frac{2 \beta \omega A}{R} + \frac{2 \beta \omega}{\nu R}(P_0 - A z) v_z\right] r \sin \omega\tau \tag{10}$$

with

$$v_z (r, z, \tau)|_{\tau=0} = 0 \tag{11}$$

$$v_z (r, z, \tau)|_{r=0} = \text{finite} \tag{12}$$

$$v_z (r, z, \tau)|_{r=R} = 0 \tag{13}$$

Equation (10) was solved for the velocity profile by taking the Laplace transform twice and then inverting to yield Equation (14)

$$v_z = 2 B \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n J_1(\mu_n)} \left[\frac{\frac{\nu \mu_n^2}{R^2} \sin \omega\tau - \omega \cos \omega\tau + \omega e^{-\nu \mu_n^2 \tau/R^2}}{\left(\frac{\nu \mu_n^2}{R^2}\right)^2 + \omega^2}\right] + 2 C \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n J_1(\mu_n)} \left[\frac{\frac{\nu \mu_n^2}{R^2} e^{-\nu \mu_n^2 \tau/R^2} - \frac{\nu \mu_n^2}{R^2}\cos \omega\tau - \omega \sin \omega\tau}{\left(\frac{\nu \mu_n^2}{R^2}\right)^2 + \omega^2}\right] + 2 C \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n J_1(\mu_n)} \left[\frac{R^2}{\nu \mu_n^2}(1 - e^{-\nu \mu_n^2 \tau/R^2})\right] \tag{14}$$

where

$$B = \frac{2 \beta \omega \nu A}{R} + \frac{2 \beta \omega}{R}(P_0 - A z) V_0 \tag{15}$$

$$C = \frac{A}{\rho} - \frac{2 \beta A V_0^2}{R} \tag{16}$$

$V_0$ = an empirical constant discussed in the Appendix. Complete details are given elsewhere (Ahn, 1970). The average flow velocity was obtained from Equations (14) and (17).

$$\langle v_z \rangle = \frac{\int_0^{\langle R \rangle} \int_0^{2\pi} \int_0^{2\pi/\omega} v_z r \, dr \, d\theta \, d\tau}{\int_0^{\langle R \rangle} \int_0^{2\pi} \int_0^{2\pi/\omega} r \, dr \, d\theta \, d\tau} \tag{17}$$

The result is

$$\langle v_z \rangle = \frac{4 \beta \omega^3 \langle R \rangle^5 A}{\pi \nu^2}\left(1 - \frac{z V_0}{\nu} + \frac{P_0 V_0}{\nu A}\right) s_1' + \frac{2 g_c \langle R \rangle^2}{\pi \nu}\left(\frac{A}{\rho} - \frac{2 \beta A V_0^2}{g_c \langle R \rangle}\right) s_2' \tag{18}$$

where

$$s_1' = \sum_{n=1}^{\infty} \frac{1 - e^{-2\pi \mu_n^2/\lambda}}{\mu_n^8 + \lambda^2 \mu_n^4} \tag{19}$$

$$s_2' = \sum_{n=1}^{\infty} \left[\frac{2\pi}{\mu_n^4} - \left(\frac{\lambda}{\mu_n^6} + \frac{\lambda}{\mu_n^6 + \lambda^2 \mu_n^2}\right)(1 - e^{-2\pi \mu_n^2/\lambda})\right] \tag{20}$$

$$\lambda = \langle R \rangle^2 \omega/\nu \tag{21}$$

$$J_0(\mu_n) = 0 \tag{22}$$

Equation (18) was then solved explicitly for the friction factor $f_p$

$$f_p = \frac{N_{Re} - \frac{8 \beta P_0 V_0}{\pi \nu} s_1}{\frac{N_{Re}^2}{\pi}\left(1 - \frac{2 \beta \rho V_0^2}{g_c \langle R \rangle}\right) s_2 - \left(\frac{2 \beta \rho \nu z N_{Re}^2 V_0}{\pi g_c \langle R \rangle^3}\right) s_1} \tag{23}$$

where

$$s_1 = \sum_{n=1}^{\infty} \frac{\lambda^3}{\mu_n^8 + \lambda^2 \mu_n^4} \tag{24}$$

$$s_2 = \sum_{n=1}^{\infty} \left[\frac{2\pi}{\mu_n^4} - \frac{\lambda}{\mu_n^6} + \frac{\lambda}{\mu_n^6 + \lambda^2 \mu_n^2}\right] \tag{25}$$

$$N_{Re} = \frac{2 \langle R \rangle \langle v_z \rangle}{\nu} \tag{26}$$

When the frequency in Equation (23) is set equal to zero, ($\lambda = 0$), we get an expression for steady state flow in a distensible (tapered) tube,

$$f_p = \frac{16}{N_{Re}\left(1 - \frac{2 \beta \rho V_0^2}{g_c \langle R \rangle}\right)} \tag{27}$$

With flow experiments and Equation (27) it was possible to evaluate $V_0$. In Equation (27), as the linear tube expansion coefficient $\beta$ approaches zero (a rigid tube), Equation (27) reduces properly to that for Poiseuille flow, $f_p = 16/N_{Re}$.

## EXPERIMENTAL PROCEDURE

The experimental friction factor can be calculated from the Fanning equation

$$f_p = \frac{\langle \Delta P \rangle \langle R \rangle g_c}{\rho z \langle v_z \rangle^2} \tag{28}$$

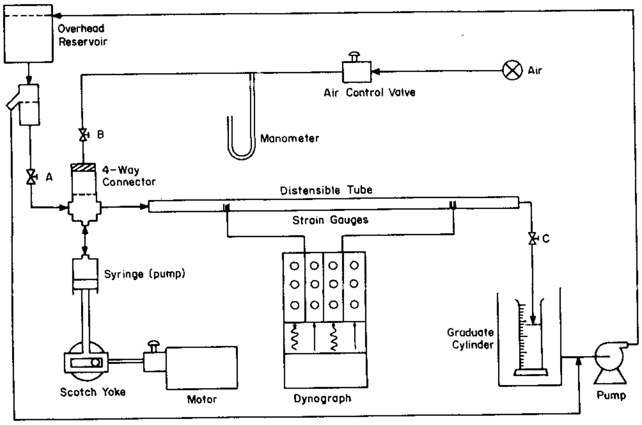


**Fig. 2. Schematic flow diagram of the equipment.**

where

$$<\Delta P> = \frac{\int_0^{2\pi/\omega} (\Delta P)\, dt}{\int_0^{2\pi/\omega} dt} \tag{29}$$

and ($\Delta P$) is the pressure drop at time $t$. The schematic flow diagram of the equipment is shown in Figure 2. It is similar to that used by Hershey and Song (1967) except that a smooth rubber tube 1.5 m long and $4.74 \times 10^{-3}$ m I.D. was used as the test section. An overhead constant head reservoir about 3 m above the equipment was used to provide a steady flow.

A sinusoidal pulse was generated by a 30-cc syringe which was connected to a scotch yoke and powered by a ⅛ horsepower variable speed motor (General Electric, 0 to 400 rev/min). An air pocket in the 4-way connector helped dampen the distortion of the pressure pulse.

The pressure drop along the tube was measured by two metal-film strain gauges of type HE 141 (Automation Industries, Inc.). The strain gauges were installed 0.8 m apart. The signals from the strain gauges were recorded on a 4-channel Dynograph (Beckman, Type R).

The extra 0.7 m of the tube length before the upstream strain gauge was designed to minimize the entrance effects. The tube length from the downstream strain gauge to the outlet was approximately 0.3 m.

The experimental procedure was similar to that described previously (Hershey and Song, 1967). The amplitude of the flow ranged from 1 to 42 cc/cycle and frequency from 0 to 1.3 cycle/s. The experiment was run at room temperature which varied from 22° to 28°C. The pressure in the tube ranged from 0 to $1.38 \times 10^4$ N/m² (gauge). The average Reynolds number ranged from 300 to 2,000. The experimental friction factor was calculated from a modified Fanning equation, as was done previously (Hershey and Song, 1967).

## COMPARISON OF EXPERIMENTAL AND PREDICTED RESULTS

Using Equation (23), the theoretical friction factor versus Reynolds number curves with $\lambda$ as parameter were calculated and are presented in Figure 3. In order to compare the theoretical friction factor with the experimental friction factor, other theoretical curves were calculated corresponding to the actual experimental conditions and are presented in Figures 4 through 8.

## DISCUSSION OF RESULTS

The theoretical and experimental results agreed within 13% up to $\lambda = 20$. Above $\lambda = 20$, the predicted results gave friction factors that were too high. For $\lambda$ greater than 20 it is believed that effects of flow acceleration and deceleration dominate over the effects of viscosity. This flow condition when $\lambda$ is greater than 20 can be compared to the unstable region for Poiseuille flow ($\lambda$ is a ratio of inertial to viscous force).

In Figure 9, it is observed that the experimental friction factor approaches a constant value at high $\lambda$ values for a given Reynolds number. This might be explained with Uchida's derived pressure loss equation for pulsatile flow (Uchida, 1956). According to Uchida the pressure drop is large at low $\lambda$ values where the viscous component dominates over the inertial component. At high $\lambda$ values the stronger inertial component influence causes smaller pressure changes.

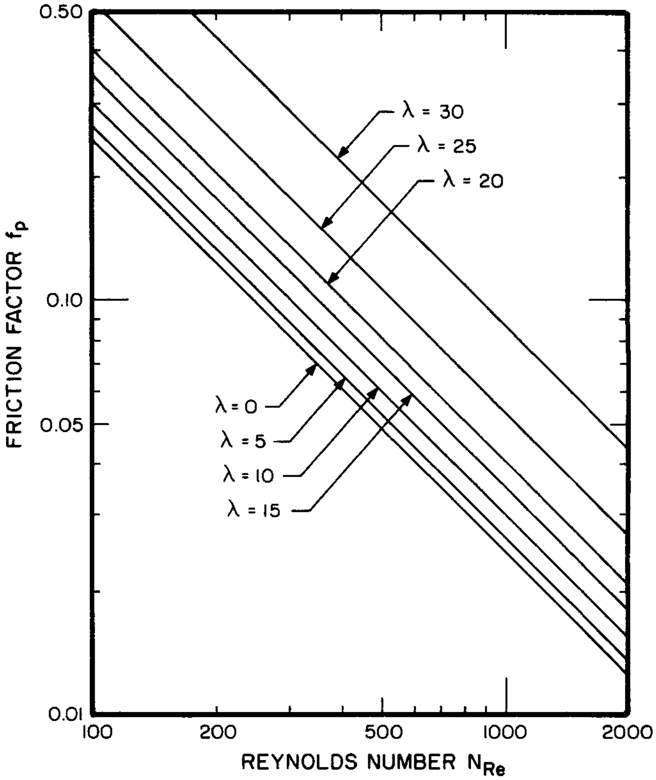


**Fig. 3. Theoretical friction factor versus Reynolds number curves for water with $\lambda$ as parameter.**

**($\beta = 4.80 \times 10^{-9}$ $m/_{N/m^2}$, $T = 25.0$°C)**

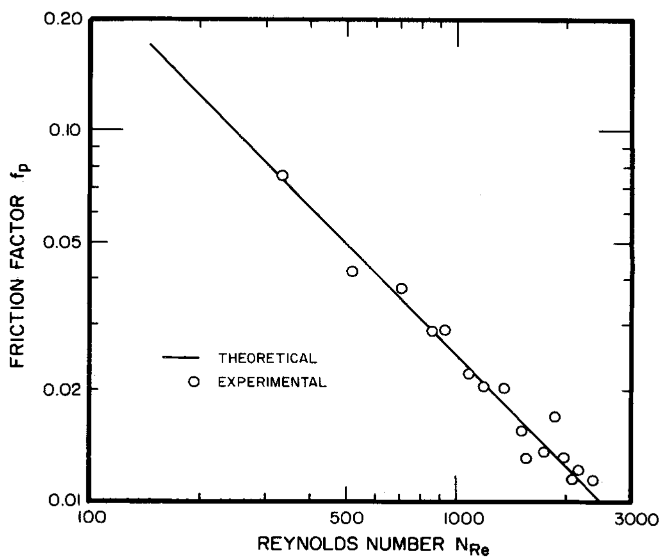


**Fig. 4. Friction factor versus Reynolds number for water at $\lambda = 0$.**

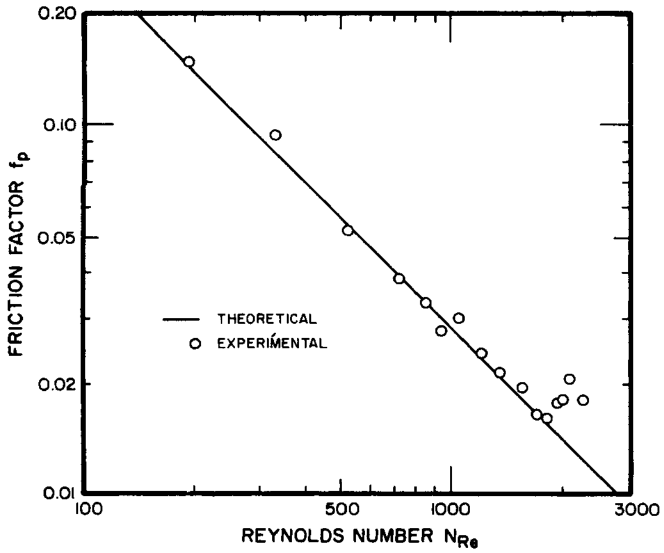


**Fig. 5. Friction factor versus Reynolds number for water at λ = 7.**

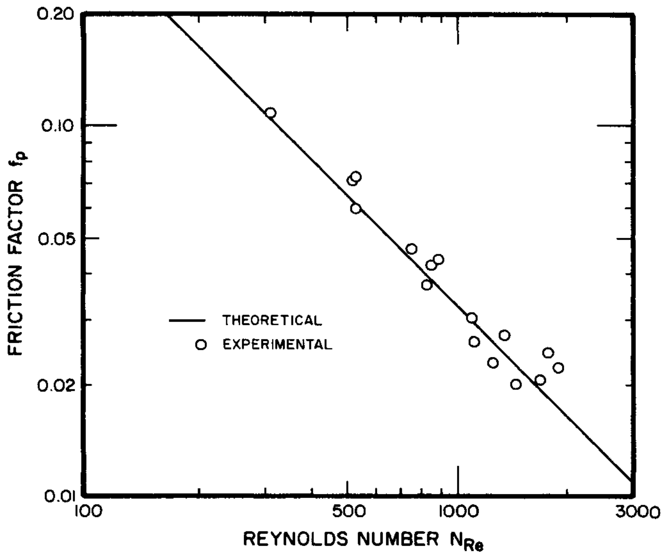


**Fig. 6. Friction factor versus Reynolds number for water at λ = 13.**

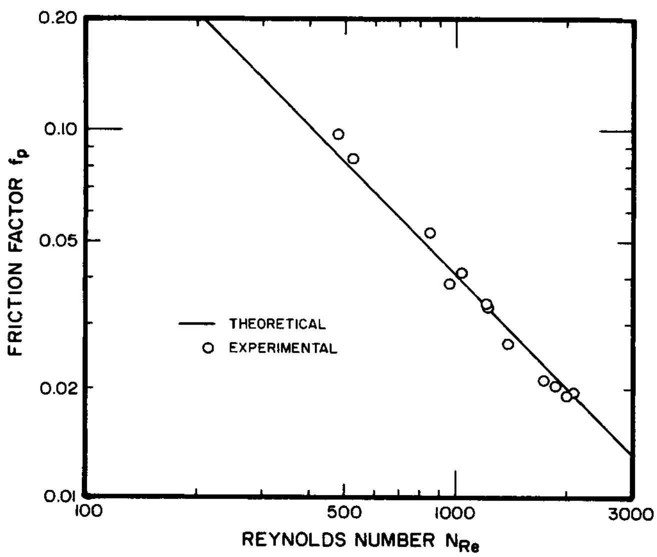


**Fig. 7. Friction factor versus Reynolds number for water at λ = 20.**

The experimental and theoretical friction factor versus Reynolds number curves for the distensible tube are parallel to but higher than those for the rigid tube results of Hershey and Song (1967). The movement of the fluid and the tube in the radial direction could be the cause of the larger energy dissipation, thus effecting larger pressure losses compared to the one-dimensional flow in the rigid tube.

It appears that the peak velocity of systole in the cardiac cycle is large enough to produce turbulence. However, from observations of injected dye in the blood stream of the aorta, McDonald (1960) indicated incomplete mixing of blood. Also he estimated λ values for the aorta and femoral arteries of man to be approximately 10 and 100, respectively. His rough estimates of λ values are questionable, since he assumed the blood to be a Newtonian fluid.

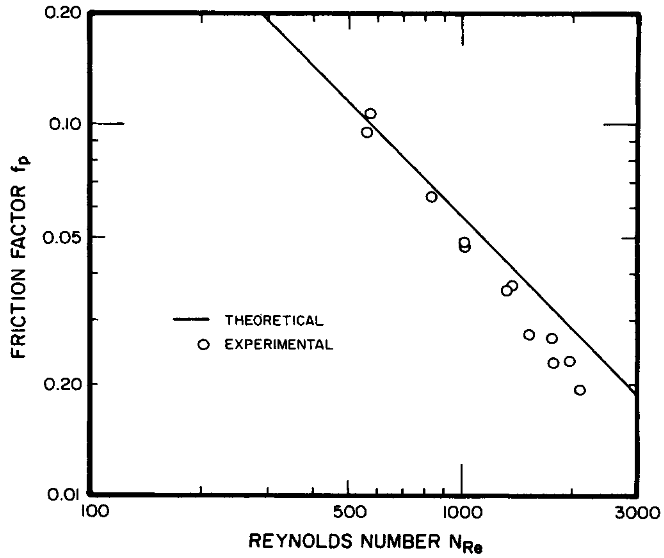


**Fig. 8. Friction factor versus Reynolds number for water at λ = 26.**

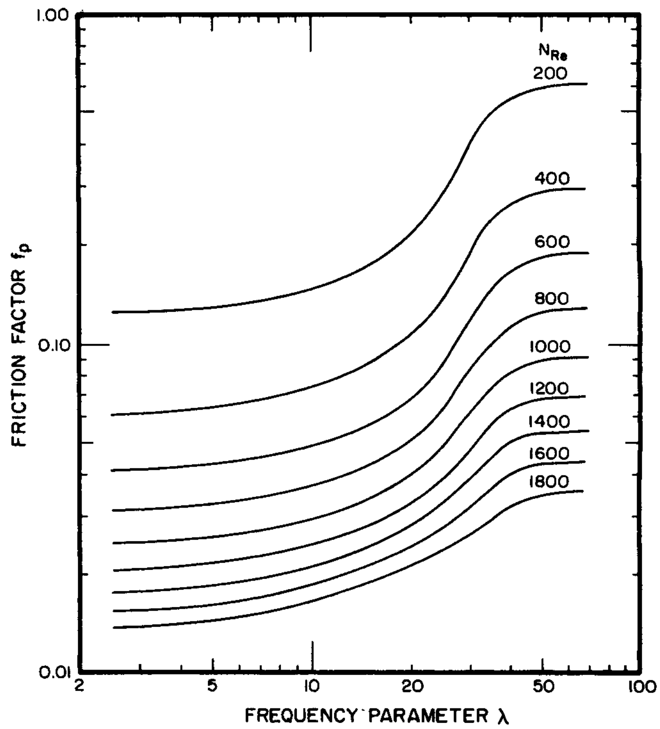


**Fig. 9. Experimental friction factor versus frequency parameter for water with Reynolds number as parameter.**

The viscosity of blood is found to be non-Newtonian (Charm et al., 1969). Allowing for some uncertainty in McDonald's values, it is nevertheless probable that the derived model reported on here is at least applicable to the femoral artery of man.

## NOTATION

$A$ = linear amplitude of pulsation
$c_1$ = constant
$D$ = diameter of the tube
$f$ = frequency
$f_p$ = friction factor for pulsatile flow
$g_c$ = 32.174 $(\text{lb}_f)\ (\text{ft})/(\text{lb}_m)\ (\text{s}^2)$
$I_0$ = modified Bessel function of the first kind of order zero
$J_0$ = Bessel function of the first kind of order zero
$L$ = Laplace transform symbol
$N_{Re}$ = Reynolds number
$P, P_0$ = pressure
$\Delta P$ = pressure difference
$<\Delta P>$ = average pressure drop with respect to time
$Q$ = average flow rate
$R$ = radius of the distensible tube
$R_0$ = radius of the distensible tube at zero stress
$<R>$ = average tube radius with respect to time and axial distance
$r$ = radial distance from axis
$t$ = time
$v_r$ = radial velocity of component
$v_z$ = axial velocity component
$<v_z>$ = average axial velocity with respect to time and velocity
$z$ = axial distance

### Greek Letters

$\beta$ = linear expansion coefficient of the distensible tube
$\theta$ = angular direction coordinate
$\lambda$ = dimensionless parameter equal to $<R>^2\ \omega/\nu$
$\mu$ = fluid viscosity
$\mu_n$ = roots of zero order Bessel function
$\nu$ = kinematic viscosity, $\mu/\rho$
$\rho$ = fluid density
$\tau$ = time defined by $\omega\tau = \omega\tau + (3\pi/2)$
$\omega$ = angular velocity, $\omega = 2\pi f$

## LITERATURE CITED


Ahn, J. S., "Determination of Friction Factors for Pulsatile Flow of Water in Distensible Tubes", Ph.D. dissertation, Univ. Cincinnati, Ohio (1970).
Atabek, H. B., and C. C. Chang, "Oscillatory Flow Near the Entry of a Circular Tube," *ZAMP*, **12**, 185 (1961).
Auster, A. M., "Digital Simulation of Pulsatile Flow in a Tapered, Elastic Tube," Ph.D. dissertation, Tulane Univ., New Orleans, Louisiana (1968).
Chang, C. C., and H. B. Atabek, "The Inlet Length for Oscillatory Flow and Its Effects on the Determination of the Rate of Flow in Arteries," *Phy. Med. Biol.*, **6**, 303 (1961).
Charm, S. E., et al., "Absence of Transition in Viscosity of Human Blood Between Shear Rates of 20 and 100 sec$^{-1}$," *J. Appl. Physiol.*, **26**, 389 (1969).
Fairchild, B. T., et al., *Chemical Engineering in Medicine and Biology*, Daniel Hershey, (ed.), pp. 3-44, Plenum Press, New York (1967).
Gilbrech, D. A., and G. D. Combs, Eng. Exp'tal Station, Univ. Arkansas, **4**, (April, 1964).
Hanks, R. W., "Velocity Profile Development in the Entrance Region of a Right Circular Cylinder with Slip at the Walls," *Phys. Fluids*, **6**, 1645 (1963).
Hershey, D., and C. S. Im, "Critical Reynolds Number for Sinusoidal Flow of Water in Rigid Tubes," *AIChE J.*, **14**, 807 (1968).
Hershey, D., and G. Song, "Friction Factors and Pressure Drop for Sinusoidal Laminar Flow of Water and Blood in Rigid Tubes," *ibid.*, **13**, 491 (1967).
Lambert, J. W., "On the Nonlinearities of Fluid Flow in Nonrigid Tubes," *J. Franklin Inst.*, **266**, 83 (1958).
McDonald, D. A., *Blood Flow in Arteries*, p. 59, The Williams & Wilkins Co., Baltimore, Md. (1960).
Morgan, G. W., and W. R. Ferrante, "Wave Propagation in Elastic Tubes Filled with Streaming Liquid," *J. Acoust. Soc. Am.*, **27**, 715 (1955).
Sarpkaya, T., "Experimental Determination of the Critical Reynolds Number for Pulsating Poiseuille Flow," *Trans. Am. Soc. Mech. Eng.*, No. 66-FE-S (1966).
Sexl, T., "Vber den von E. G. Richardson entdehten 'Annulareffekt'," *Z. Phys.*, **61**, 349 (1930).
Sheppard, R. G., "The Flow of Fluids Through Expansible Tubes," Ph.D. dissertation, Univ. Iowa, Iowa City (1967).
Streeter, V. L., W. F. Keitzer, and D. F. Bohr, "Pulsatile Pressure and Flow through Distensible Vessels," *Circulation Res.*, **13**, 3 (1963).
Taylor, A. E., *Advanced Calculus*, 51, Ginn, New York (1955).
Uchida, S. "The Pulsating Viscous Flow Superposed on the Steady Laminar Motion of Incompressible Fluid in a Circular Pipe," *ZAMP VII*, 403 (1956).
Womersley, J. R., *The Mathematical Analysis of the Arterial Circulation in a State of Oscillatory Motion*, Tech. Rept. WADC TR 56-614 (1957).
Yellin, E. L., *Dissipation of Turbulence in Oscillatory Tube-Flow: The Relaxation Time*, ASME Publication 66-WA/BHF-1 (1966).


## APPENDIX

### Laplace Transformation of the Nonlinear Term in Equation (10)

The Laplace transform of the nonlinear term $\frac{2\beta A}{\nu R} r\, v_z^2 \cos \omega\tau$ in Equation (10) is shown below

$$\underset{\tau\to s}{L} \frac{2A\beta}{\nu R} r\, v_z^2 \cos\omega\tau = \frac{2A\beta r}{\nu R}\int_0^\infty e^{-s\tau} v_z^2 \cos\omega\tau\, d\tau \tag{A1}$$

By using a variation of the mean value theorem (Taylor, 1955), the term $v_z^2$ in Equation (A1) was taken out of the integral sign.

$$\underset{\tau\to s}{L} \frac{2A\beta r}{\nu R} v_z^2 \cos\omega\tau \cong \frac{2A\beta r}{\nu R} v_z^2(\tau_m, r)\int_0^\infty e^{-s\tau}\cos\omega\tau\, d\tau \tag{A2}$$

where $\tau_m$ is some intermediate value of the range of $\tau$. The result is

$$\underset{\tau\to s}{L} \frac{2A\beta r}{\nu R} v_z^2 \cos\omega\tau \cong \frac{2A\beta r}{\nu R} v_z^2(\tau_m, r)\frac{s}{s^2+\omega^2} \tag{A3}$$

If now a second Laplace transform is taken with respect to $r$, $\underset{r\to p}{L}$, the result is Equation (A4)

$$\underset{r\to p}{L} \frac{2A\beta}{\nu R}\frac{s}{s^2+\omega^2} r\, v_z^2(\tau_m, r) \cong \frac{2A\beta}{\nu R}\frac{s}{s^2+\omega^2}\frac{1}{p^2} v_z^2(\tau_m, r_m) \tag{A4}$$

where $r_m$ is some intermediate value of the range of $r$. In this paper $v_z^2(r_m, \tau_m)$ is simply expressed as $V_0^2$. Steady flow experimental data ($\omega = 0$) applied to Equation (27) enabled us to evaluate $V_0$ and verify that $V_0$ was actually a constant.

*Manuscript received June 15, 1972; revision received August 21, 1972; paper accepted August 22, 1972.*